

PHYS5150 — PLASMA PHYSICS  
LECTURE 22 - ELECTROSTATIC WAVES IN COLD  
MAGNETIZED PLASMAS

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1 ELECTROSTATIC WAVES IN COLD MAGNETIZED PLASMAS

- Why cold? Initial interest was for ionospheric science
- Consider both, electron and ion motion
- Equations:
  - Poisson equation
  - Continuity equation
  - 3-D momentum equation with  $p = 0$  (cold ionosphere) and  $B_0 \neq 0$
- Geometry: Wave might have  $\mathbf{k} \parallel \mathbf{B}$ ,  $\mathbf{k} \perp \mathbf{B}$ , or in-between

Without loss of generality we chose  $\mathbf{B}$  and  $\mathbf{k}$  such that

$$\mathbf{B} = [0, 0, B_z]$$

$$\mathbf{k} = [k_x, 0, k_z].$$

We consider electrostatic waves, which implies that  $\mathbf{k} \parallel \mathbf{E}$ , and consequently

$$\mathbf{E}(\mathbf{x}, t) = [E_x, 0, E_z] e^{i(k_x x + k_z z - \omega t)}.$$

We note further that

$$\mathbf{E} = -\nabla\phi = [-ik_x\phi, 0, -ik_z\phi]$$

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and

$$\nabla \cdot \mathbf{E} = -\nabla^2 \phi = k_x^2 \phi + k_z^2 \phi.$$

The Poisson equation gives

$$\nabla^2 \phi = -\frac{\rho}{\epsilon_0} = -\frac{1}{\epsilon_0} \sum_{s=i,e} \delta n_s q_s,$$

the continuity equation reads

$$\begin{aligned} \frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \mathbf{u}_s) &= 0 \\ -i\omega \delta n_s + n_0 i(k_x u_x + k_z u_z) &= 0, \end{aligned}$$

and the momentum equation writes as

$$m_s n_s \left( \frac{\partial \mathbf{u}_s}{\partial t} + \underbrace{(\mathbf{u}_s \cdot \nabla) \mathbf{u}_s}_{O(u_s^2) \sim 0} \right) = \underbrace{-\nabla p}_{T=0 \rightarrow 0} + n_s q_s \mathbf{E} + n_s q_s \mathbf{u}_s \times \mathbf{B}.$$

We first solve the  $\hat{\mathbf{x}}$  component of the momentum equation to find  $u_{x,s}$  as function of  $u_{y,s}$ :

$$-i\omega m_s u_{x,s} = q_s E_x + q_s u_{y,s} B_z,$$

then solve then the  $\hat{\mathbf{y}}$  component to obtain a second relation for these velocities

$$\begin{aligned} -i\omega m_s u_{y,s} &= -q_s u_{x,s} B_z \\ u_{y,s} &= -\frac{i q_s u_{x,s} B_z}{\omega m_s} = -i \frac{\omega_{c,s}}{\omega} u_{x,s}, \end{aligned}$$

and substitute  $u_{y,s}$  into the  $\hat{\mathbf{x}}$  component

$$\begin{aligned} -i\omega m_s u_{x,s} &= q_s E_x + \underbrace{q_s B_z}_{m_s \omega_{c,s}} u_{y,s} \quad \left| \leftarrow u_{y,s} \right. \\ &= q_s E_x - i \frac{m_s}{\omega} \omega_{c,s}^2 u_{x,s} \\ -i\omega^2 u_{x,s} &= \frac{q_s}{m_s} E_x \omega - i\omega_{c,s}^2 u_{x,s} \\ u_{x,s} &= i \frac{q_s}{m_s} E_x \left( \frac{\omega}{\omega^2 - \omega_{c,s}^2} \right). \end{aligned}$$

Finally, we obtain the relation for  $u_{z,s}$  from the  $\hat{\mathbf{z}}$  component:

$$-i\omega m_s u_{z,s} = q_s E_z$$

$$u_{z,s} = i \frac{q_s}{m_s} E_z \frac{1}{\omega}.$$

We now put velocities into the continuity equation and yield

$$0 = -i\omega \delta n_s + in_0 \left( i \frac{q_s}{m_s} k_x E_x \frac{\omega}{\omega^2 - \omega_{c,s}^2} + i \frac{q_s}{m_s} k_z E_z \frac{1}{\omega} \right).$$

We use that  $\mathbf{E} = -\nabla\phi$ , implying that  $E_x = -ik_x\phi$  and  $E_z = -ik_z\phi$ . After rearranging the equation above we get an expression for  $\delta n_s$

$$\delta n_s = n_0 \frac{q_s}{m_s} \left[ \frac{k_x^2}{\omega^2 - \omega_{c,s}^2} + \frac{k_z^2}{\omega^2} \right] \delta\phi,$$

which we put into the Poisson equation

$$(k_x^2 + k_z^2) \delta\phi = \frac{1}{\epsilon_0} \sum_{s=i,e} \delta n_s q_s$$

$$= \delta\phi \sum_{s=i,e} \underbrace{\left( \frac{n_0 q_s^2}{\epsilon_0 m_s} \right)}_{\omega_{p,s}^2} \left[ \frac{k_x^2}{\omega^2 - \omega_{c,s}^2} + \frac{k_z^2}{\omega^2} \right] \quad \Big| \cdot \frac{\omega^2}{k^2},$$

and after rearranging the equation for  $\omega$  we get

$$\boxed{\omega^2 = \sum_{s=i,e} \frac{\omega_{p,s}^2}{k^2} \left[ k_z^2 + \frac{k_x^2}{1 - \frac{\omega_{c,s}^2}{\omega^2}} \right]} \quad (1)$$

The resulting *dispersion relation for electrostatic waves propagating in a cold plasma* is rather complicated, but very useful, albeit only in simplified limits.

### 1.1 Dispersion relation for $\mathbf{B} \rightarrow 0$

This case implies that  $\omega_{c,s} \rightarrow 0$ , which recovers the *plasma oscillation case*

$$\boxed{\omega^2 = \omega_{pe}^2 + \omega_{pi}^2}. \quad (2)$$

### 1.2 Dispersion relation for $\mathbf{k} \parallel \mathbf{B}$

In this case there is no  $\delta\mathbf{u} \times B_z$  force acting on the plasma particles. Thus, this scenario is similar to the previous one and we get again

$$\boxed{\omega^2 = \omega_{pe}^2 + \omega_{pi}^2} \quad (3)$$

### 1.3 Strongly magnetized plasma

Strongly magnetized means that  $\mathbf{B} \rightarrow \infty$ , and therefore is  $\omega_{c,s} \gg \omega$ ,

$$\frac{k_x^2}{1 - \frac{\omega_{c,s}^2}{\omega^2}} \sim O\left(\frac{\omega}{\omega_{c,s}}\right)^2 \approx 0,$$

and the dispersion relation becomes

$$\omega^2 = (\omega_{pe}^2 + \omega_{pi}^2) \frac{k_z^2}{k^2}.$$

After replacing  $\frac{k_z^2}{k^2}$  by  $\cos \theta$ , where  $\theta$  is the angle between  $\mathbf{B}$  and  $\mathbf{k}$ , the dispersion relation for a strongly magnetized plasma writes as

$$\boxed{\omega^2 = (\omega_{pe}^2 + \omega_{pi}^2) \cos \theta} \quad (4)$$